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Global Attractivity and Forward Neural Networks

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Abstract—Triangular dynamical systems can be used to model neural networks of forward type (FNN). In this paper, we establish some convergence theorem for such systems, which indicate how FNN should be implemented to perform the task for which they are designed.

Keywords—Triangular systems, Neural networks, Orbits, Stationary state, Global attractors.

1. INTRODUCTION

The interest in the theory and applications of Neural Networks (NN) has increased dramatically in recent years [1–3]. Structured like a human brain, with different levels of interconnected neurons, they have opened many new frontiers in different areas of science and technology. Forward networks, in which the layers of neurons are hierarchically arranged and the connections never go backward, can be modeled by discrete dynamical systems of the triangular type [4]. Therefore, the study of such systems is of significant help in understanding the behavior of this class of networks. The purpose of our paper is to show that a large family of systems of triangular type admits an equilibrium point which is a global attractor. Accordingly, a forward neural network, which can be modeled by a triangular system of this family, will converge to the equilibrium regardless of the state of the net at the beginning of the process (initial state). To be more specific, consider the discrete Hopfield [5] model of a neural network with N neurons. Its formulation as a dynamical system if $\mathbf{R}^N$ takes the form

$$\mathbf{x}_{n+1} = (I - C)\mathbf{x}_n + TF(\mathbf{x}_n) + \mathbf{x}_J. \tag{1.1}$$

The components of the vector $\mathbf{x}_n$ represent the energy level of the various neurons of the network at time n. C is the *leakage* matrix, an $N \times N$ diagonal matrix accounting for the energy lost by the system at each iteration. T is the *connectivity* matrix, with its entry $t_{ij}, 1 \leq i, j \leq N$, representing the excitatory (> 0) or inhibitory (< 0) action of neuron j on neuron i. $F(\mathbf{x}) = (f_1(x_1), \ldots, f_N(x_N))$ is the neuron response function, which provides a threshold level below which the neurons are inactive. $\mathbf{x}_j$ is an N-dimensional input vector. The neurons of the network are usually divided into input, hidden and output layers. Accordingly, the entries of the N-dimensional vector $\mathbf{x}$ representing the neurons are organized so that the input neurons come first, followed by the hidden and then by the output neurons. A network modeled by (1.1) is said to be forward if

$$t_{ij} = 0 \qquad \text{for } j \geq i. \tag{1.2}$$

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Consequently, the k^{th} equation of (1.1) assumes the form

$$x_{k,n+1} = (1-c_k)x_{k,n} + t_{k1}f_1(x_{1,n}) + \cdots + t_{k(k-1)}f_{k-1}(x_{k-1,n}) + x_{kJ} \tag{1.3}$$

and the dynamical system (1.1) is said to be triangular. This type of NN arises, for example, in those cases in which the entries of the connectivity matrix are selected using learning rules like back-propagation [6]. Our goal is to prove some global convergence theorems for this type of networks (see also [7,8]).

The paper is organized as follows: Section 2 contains some notations and terminology used throughout; in Section 3 we establish our results; and Section 4 illustrates with some examples the sharpness of the assumptions under which our theorems are proved.

2. NOTATIONS AND DEFINITIONS

Let X be a region of $\mathbf{R}^q$. A function $F : X \to X$ defines the discrete dynamical system

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n), \tag{2.1}$$

from which the state of the system at time $n+1$, $\mathbf{x}_{n+1}$, is derived, when its state at time $n, \mathbf{x}_n$, is known. The *orbit* of a point $\mathbf{x}_0 \in X$ is the sequence of states $O(\mathbf{x}_0) = \{\mathbf{x}_0, \mathbf{x}_1 = F(\mathbf{x}_0), \dots, \mathbf{x}_{n+1} = F(\mathbf{x}_n), \dots\}$. A point $\mathbf{x}_s \in X$ is a *stationary* (or *equilibrium* or *fixed*) point of the dynamical system governed by F if $O(\mathbf{x}_s) = \{\mathbf{x}_s\}$. A *periodic* orbit of *prime period* p, usually denoted by $O(\mathbf{x}_0, p)$, is an orbit such that

$$\mathbf{x}_p = \mathbf{x}_0 \quad \text{and} \quad \mathbf{x}_k \neq \mathbf{x}_0 \qquad \text{for all } k < p. \tag{2.2}$$

Notice that every state of $O(\mathbf{x}_0, p)$ is a fixed point of the p^{th} iterate F^p of F. An equilibrium point $\mathbf{x}_s$ is said to be a *global attractor* if every orbit $O(\mathbf{x}_0)$ converges to $\mathbf{x}_s$. A point $\mathbf{z} \in X$ is a limit point of $O(\mathbf{x}_0)$ if there is a subsequence of the orbit which converges to $\mathbf{z}$. The set of limit points of $O(\mathbf{x}_0)$ is denoted by $L(\mathbf{x}_0)$. In the case when F is continuous and the orbit $O(\mathbf{x}_0)$ is bounded it is easy to show that $L(\mathbf{x}_0)$ is a nonempty, closed and bounded set, and has the important property

$$F(L(\mathbf{x}_0)) = L(\mathbf{x}_0). \tag{2.3}$$

Let U and V be two open subsets of X and $F : U \to U$, $G : V \to V$ be two continuous functions. Assume that there exists $H : U \to V$, continuous, onto, and with continuous inverse such that

$$H(F(\mathbf{x})) = G(H(\mathbf{x})) \qquad \text{for every } \mathbf{x} \in U. \tag{2.4}$$

Then F and G are said to be *topologically conjugate* (by H). Notice that two discrete dynamical systems which are topologically conjugate display the same behavior. In particular, topological conjugacy preserves global attractivity. A map $F : \mathbf{R}^q \to \mathbf{R}^q$ is said to be *lower triangular* if

$$F(\mathbf{x}) = F(x_1, \dots, x_q) = (f_1(x_1),\, f_2(x_1, x_2) \dots, f_q(x_1, \dots, x_q)). \tag{2.5}$$

Upper triangular maps are defined analogously.

3. RESULTS

In this section we shall obtain the following convergence theorems for discrete dynamical systems governed by triangular functions.

THEOREM 1. *(See [4].) Let F be a continuous, lower triangular map and let $\mathbf{x}_s = (x_{s1}, \dots, x_{sq})$ be a fixed point of F. Assume that*

(1) *there exists $k \in (0,1)$ such that $|f_1'(x_1)| \leq k$;*

(2) *there exists $r > 0$ such that $|x_i - x_{si}| \leq r$, $i = 1, 2, \dots, j$ implies $|\frac{\partial f_{j+1}}{\partial x_{j+1}}| \leq k$.*

Then $\mathbf{x}_s$ is the only equilibrium point of F and every orbit converges to it.

THEOREM 2. *Let F be a continuous, lower triangular map and let $\mathbf{x}_s = (x_{s1}, \dots, x_{sq})$ be a fixed point of F. Assume that*

(1) $|f_1'(x_1)| \le 1$ *and there exists $r > 0$ such that $|f_1'(x_1)| < 1$ if $0 < |x_1 - x_{s1}| \le r$;*
(2) $|(\frac{\partial f_i}{\partial x_i})(0, \dots, x_i)| \le 1$, *and* $|(\frac{\partial f_i}{\partial x_i})(0, \dots, x_i)| < 1$ *if* $0 < |x_i - x_{si}| \le r$, $i = 2, \dots, q$;
(3) *there exists $M > 0$ such that $|\frac{\partial f_j}{\partial x_i}| \le M$, for $i < j$ and $|x_i - x_{si}| \le r$.*

Then $\mathbf{x}_s$ is the only equilibrium point of F and every orbit converges to it.

THEOREM 3. *Let F be a continuous, lower triangular map and let $\mathbf{x}_s = (x_{s1}, \dots, x_{sq})$ be a fixed point of F. Assume that*

(1) $|f_1'(x_1)| \le 1$ *and there exists $r > 0$ such that $|f_1'(x_1)| < 1$ if $0 < |x_1 - x_{s1}| \le r$;*
(2) $|x_i - x_{si}| \le r$, $i = 1, \dots, j-1$ *implies* $|\frac{\partial f_j}{\partial x_j}| \le 1$ *and* $|\frac{\partial f_j}{\partial x_j}| < 1$ *if, in addition,* $0 < |x_j - x_{sj}| \le r$.

Then $\mathbf{x}_s$ is the only equilibrium point of F and every orbit converges to it.

We see that Theorem 1 is a particular case of Theorem 3. However, we include it explicitly since its proof is significantly simpler. We begin with two lemmas which are used in the proofs of the three theorems.

LEMMA 1. *Let $F : \mathbf{R}^q \to \mathbf{R}^q$ and consider the dynamical system $\mathbf{x}_{n+1} = F(\mathbf{x}_n)$. Assume that $F(\mathbf{x}_s) = \mathbf{x}_s$. Then F is topologically equivalent to a map G such that $G(\mathbf{0}) = \mathbf{0}$.*

PROOF. Let $H(\mathbf{x}) = \mathbf{x} + \mathbf{x}_s$ and let $G(\mathbf{x}) = F(\mathbf{x} + \mathbf{x}_s) - \mathbf{x}_s$. Then $F(H(\mathbf{x})) = F(\mathbf{x} + \mathbf{x}_s)$ and $H(G(\mathbf{x})) = H(F(\mathbf{x} + \mathbf{x}_s) - \mathbf{x}_s) = F(\mathbf{x} + \mathbf{x}_s)$. Hence F and G are topologically conjugate. Moreover $G(\mathbf{0}) = \mathbf{0}$. ∎

LEMMA 2. *Let $\{a_n\}$, $\{b_n\}$ be two sequences of positive numbers. Assume that $a_n \to 0$ and there exists $k \in (0,1)$ such that $b_{n+1} \le a_n + kb_n$. Then $b_n \to 0$.*

PROOF. The sequence $\{b_n\}$ is bounded. This follows from the inequality

$$b_{n+1} \le a_n + ka_{n-1} + \cdots + k^{n-1}a_0 + k^n b_0.$$

Let L be the maximum limit of $\{b_n\}$ and denote by $\{b_{n,k+1}\}$ the subsequence of $\{b_n\}$ which converges to L. Since the maximum limit of $\{b_{n,k}\}$ cannot exceed L we have, from the inequality of the lemma, $L \le kL$. Hence $L = 0$. ∎

PROOF OF THEOREM 1. The (existence and) uniqueness of the equilibrium point follows from the fact that each function $f_i(0, 0, \dots, x_i)$ is contraction. Using conjugacy (see Lemma 1) we can assume that the equilibrium point is $\mathbf{0}$. It remains to show that every orbit converges to $\mathbf{0}$. Let $\mathbf{x}_0 = (x_{01}, \dots, x_{0q})$ be given. Obviously $x_{n1} \to 0$ as $n \to \infty$. Therefore, we may assume, without loss of generality, $|x_{n1}| \le r$ for every n. Using the Mean Value Theorem, we can now write $f_2(x_{n1}, x_{n2}) = f_2(x_{n1}, 0) + \frac{\partial f_2}{\partial x_2}(x_{n1}, tx_{n2})x_{n2}$. Consequently $|x_{(n+1)2}| \le a_n + k|x_{n2}|$, with $a_n = |f_2(x_{n1}, 0)|$. Since $a_n \to 0$ we can use Lemma 2 to obtain that $x_{n2} \to 0$. This argument can be repeated for all components. Hence the orbit converges to $\mathbf{0}$. ∎

Using the same argument as in Theorem 1, we can assume, without loss of generality, that in Theorems 2 and 3 the fixed point $\mathbf{x}_s$ is the origin. We now prove both theorems for the case when $q = 1$. Consider the function $g(x) = x - f(x)$. Since $g'(x) \ge 0$, the function is nondecreasing, and it is strictly increasing in the interval $[-r, r]$. Hence, g vanishes only at $x = 0$, which implies that 0 is the only fixed point of f. From $f(x) = f'(c)x$, we derive $|f(x)| < |x|$ if $0 < |x| \le r$. Since $|f'(x)| \le 1$ we obtain, from the previous inequality, $|f(x)| < |x|$ for very $x \in \mathbf{R}$, $x \ne 0$. Consequently, given $x_0 \ne 0$, the sequence $\{|x_n|\}$ is strictly decreasing. Let x_L be its limit. If $x_L > 0$, then neither x_L nor $-x_L$ can be fixed points of f. From the equality $f(L(x_0)) = L(x_0)$ we derive $f(x_L) = -x_L$ which contradicts the inequality $|f(x)| < |x|$ for $|x| > 0$. Hence $x_L = 0$.

PROOF OF THEOREM 2. We first present the proof for the case $q = 2$. The variables will be denoted with x and y. We want to show that given $d < r$ there exists $\delta(d) < d/2$ and $r_1 \le r$

such that $|f_2(x,y)| \le |y| - \delta(d)$ for every $|y| \ge d$ and $|x| \le r_1$. We know that $|f_2(0,\pm d)| < d$. Let $\rho(d) = \min\{d - |f_2(0,d)|,\, d - |f_2(0,-d)|,\, d/2\}$ and assume that $y \ge d$. From $f_2(0,y) = f_2(0,y) - f_2(0,d) + f_2(0,d)$ we derive $|f_2(0,y)| \le y - d + d - \rho(d) = y - \rho(d)$. Similarly if $y \le -d$ we obtain $|f_2(0,-d)| \le |y+d| + d - \rho(d) = |y| - \rho(d)$. Now

$$f_2(x,y) = f_2(0,y) + f_2(x,y) - f_2(0,y) = f_2(0,y) + \frac{\partial f_2}{\partial x}(tx,y)x.$$

Choose $r_1 \le r$ such that $Mr_1 < \rho(d)/2$. Then with $|x| \le r_1$ and $|y| \ge d$, we have

$$|f_2(x,y)| \le |y| - \rho(d) + \rho(d)/2 = |y| - \rho(d)/2 = |y| - \delta(d)$$

with $\delta(d) = \rho(d)/2$.

Let $\mathbf{x}_0 = (x_0, y_0)$ be given. Since $x_n \to 0$, we may assume that $|x_n| \le r_1$. As long as $|y_n| \ge d$, we have $|y_{n+1}| \le |y_n| - \delta(d)$. Therefore the sequence $\{x_n, y_n\}$ is bounded. Since $x_n \to 0$, the limit points of $\{x_n, y_n\}$ are of the form $(0,y)$. From the equalities $F(L(\mathbf{x}_0)) = L(\mathbf{x}_0)$, $F(0,y) = (0, f_2(0,y))$ and from the inequality $|f_2(0,y)| < |y|$, if $|y| \ne 0$, we derive $y = 0$.

It is easy to see that the above reasoning can be extended to the case $q > 2$. ∎

From the proof of Theorem 2 we understand that its assumptions can be slightly relaxed. For example, in the case $q = 3$, we can replace $|\frac{\partial f_3}{\partial y}(x,y,z)| \le M$ with $|\frac{\partial f_3}{\partial y}(0,y,z)| \le M$.

PROOF OF THEOREM 3. Once again we shall first prove the result in the case $q = 2$ and denote the variables with x and y. Since $x_n \to 0$, we may assume that $|x_n| \le r$ for all n. We also know, from the proof of Theorem 2, that given $d < r$ the exists $\delta(d)$ such that for every $|y| \ge d$, one has $|f_2(0,y)| \le |y| - \delta(d)$. Let us show that a similar result holds for all $x \in [-r,r]$. Let $x_0 \in [-r,r]$. First we show that there exists $y(x_0)$ such that $f_2(x_0, y(x_0)) = y(x_0)$ and $y(x_0) \to 0$ as $x_0 \to 0$. We know that $f_2(0,r) < r$ and $f_2(0,-r) > -r$. Let $\alpha = (1/2)\min\{r - f_2(0,r),\, f_2(0,-r) + r\}$. By the uniform continuity of f_2 in $[-r,r] \times [-r,r]$ we can find β such that

$$\|(x,y) - (u,v)\| \le \beta \quad \text{implies} \quad |f_2(x,y) - f_2(u,v)| \le \alpha.$$

Let $\beta \le r$ be as above and consider $x \in [-\beta, \beta]$. We have

$$\begin{aligned} f_2(x,r) &= f_2(0,r) + f_2(x,r) - f_2(0,r) \le f_2(0,r) + |f_2(x,r) - f_2(0,r)| \\ &\le f_2(0,r) + \frac{1}{2}(r - f_2(0,r)) < r. \end{aligned}$$

Similarly

$$\begin{aligned} f_2(x,-r) &= f_2(0,-r) + f_2(x,-r) - f_2(0,-r) \ge f_2(0,-r) - |f_2(x,-r) - f_2(0,-r)| \\ &\ge f_2(0,-r) - \frac{1}{2}(r + f_2(0,-r)) > -r. \end{aligned}$$

Consequently, the function $h_x(y) = y - f_2(x,y)$ changes sign in $[-r,r]$ and it must have a zero. This 0 is unique since h_x is strictly increasing in $[-r,r]$. Therefore, for every $x \in [-\beta,\beta]$ there is a unique $y(x) \in (-r,r)$ such that $f_2(x, y(x)) = y(x)$.

Now we prove that $y(x) \to 0$ as $x \to 0$. Let $x_n \to 0$. Assume that $y(x_n)$ does not converge to 0. Since $y(x_n) \in [-r,r]$, it has a convergent subsequence, say $y(x_{n,k}) \to y_0$, with $y_0 \ne 0$. From $f_2(x_{n,k}, y(x_{n,k})) = y(x_{n,k})$ and from the continuity of f_2 we derive $f_2(0,y_0) = y_0$, against the fact that $y = 0$ is the only fixed point of $f_2(0,y)$.

Let $d < r$ be given. We can find $b \le \beta$ such that $y(x) \in (-d/3, d/3)$ for every $|x| \le b$. Thus, using the same reasoning of Theorem 2, we can determine $\delta(d) \le d/2$, independent of $x \in (-b,b)$ such that

$$|f_2(x,y) - f_2(x, y(x))| \le |y - y(x)| - \delta(d)$$

for every $|y| \ge d$.

Let $\mathbf{x}_0 = (x_0, y_0)$. Since $x_n \to 0$, we may assume, without loss of generality, that $|x_n| \le b$ for all n. We may also assume that $|y(x_i) - y(x_j)| \le \delta(d)/2$ for every $x_i, x_j \in (-b, b)$ since $y(x) \to 0$ as $x \to 0$. Hence

$$|y_1 - y(x_0)| \le |y_0 - y(x_0)| - \delta(d)$$

and, in general,

$$|y_{n+1} - y(x_n)| \le |y_n - y(x_n)| - \delta(d)$$

as long as $|y_n| \ge d$. This implies

$$\begin{aligned} |y_{n+1} - y(x_n)| &\le |y_n - y(x_{n-1})| + |y(x_{n-1}) - y(x_n)| - \delta(d) \\ &\le |y_n - y(x_{n-1})| - \frac{\delta(d)}{2} < |y_n - y(x_{n-1})|. \end{aligned}$$

Therefore the sequence $\{x_n, y_n\}$ is bounded. Since $x_n \to 0$ the limit points of $\{x_n, y_n\}$ are of the form $(0, y)$. From the equalities $F(L(\mathbf{x}_0)) = L(\mathbf{x}_0)$, $F(0, y) = (0, f_2(0, y))$ and from the inequality $|f_2(0, y)| < |y|$ if $|y| \ne 0$, we derive $y = 0$.

The result can be extended to $\mathbf{R}^q$ with $q > 2$ with a straightforward argument. ∎

Forward neural networks modeled by (1.1) satisfy the conditions of Theorem 3 as long as $c_k \in (0, 1)$ for all $k = 1, \dots, N$. The only assumption which needs verification is the existence of a fixed point. The neuron response function F has the property $\|F(\mathbf{x})\| \le \sqrt{N}$. Hence, the function TF is bounded. Moreover, with $r = \max\{1 - c_k : k = 1, \dots, N\} < 1$ we have $\|(I - C)\mathbf{x}\| \le r\|\mathbf{x}\|$. Consequently there exists $R > 0$ such that

$$\|(I - C)\mathbf{x} + TF(\mathbf{x}) + \mathbf{x}_j\| \le R \tag{3.1}$$

for every $\|\mathbf{x}\| \le R$. Brower's fixed point theorem (see, for example, [9]) ensures the existence of a fixed point $\mathbf{x}_s$. According to Theorem 3, all orbits converge to $\mathbf{x}_s$. The fixed point changes with the input vector $\mathbf{x}_J$. More precisely, if we denote with $\mathbf{x}_K$ another input vector and with $\mathbf{x}_{Js}$, $\mathbf{x}_{Ks}$ the corresponding stationary states, we have (see [4])

$$\|\mathbf{x}_{Js} - \mathbf{x}_{Ks}\| \ge \frac{\|\mathbf{x}_J - \mathbf{x}_K\|}{\|C\| + \|T\|}. \tag{3.2}$$

Therefore, different inputs will produce different equilibria, making it possible for the neural network to perform the tasks for which it was designed.

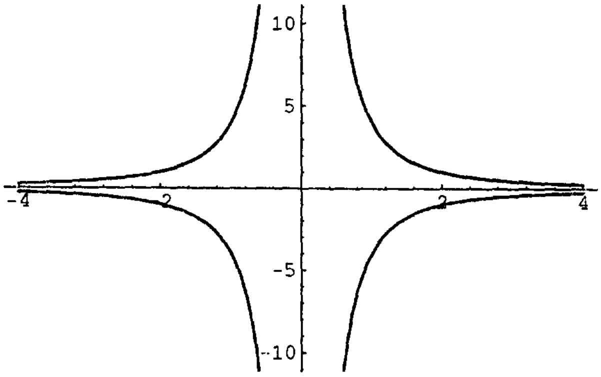
4. EXAMPLES

The following examples show that the properties assumed in Theorems 1–3 of the previous section are quite sharp.

EXAMPLE 1. Let $F(x, y) = (f_1(x), f_2(x, y)) = (0.5x, (xy)^2)$. We see that the origin is the only fixed point of F and $(\frac{d}{dx})f_1(x) = 0.5$ and $(\frac{\partial}{\partial y})f_2(0, y) = 0$. Hence, Properties (1) and (2) of Theorem 2 are verified. However, $|(\frac{\partial}{\partial x})f_2(x, y)|$ is not bounded for $x \ne 0$, no matter how close $|x|$ is to 0. Hence the third condition of Theorem 2 is not verified. The orbit of a point (x, y) with $x \ne 0$ is of the form

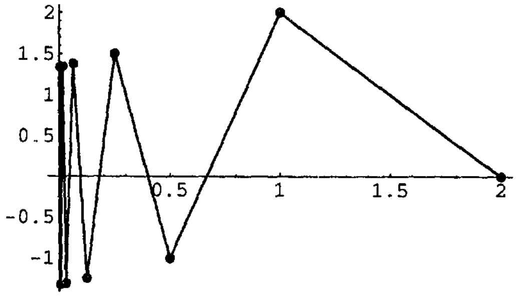
$$\mathbf{x}_n = \left(\frac{x}{2^n}, \frac{2^{2n+2}}{x^2}\left(\frac{x^2 y}{4}\right)^{2^n}\right).$$

Therefore, the orbit goes to 0 if and only if $|y| < 4/x^2$ (see the following graph).



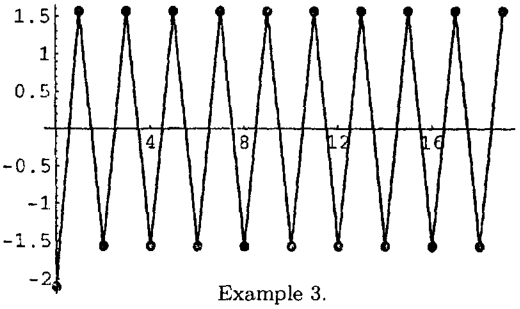
Example 1.

EXAMPLE 2. Let $F(x, y) = (0.5x, x - y)$. Notice that $(0,0)$ is an equilibrium point and the first condition of Theorem 3 is verified. For the second condition we have $|(\frac{\partial}{\partial y})f_2(x,y)| = 1$. Therefore the property "$|\frac{\partial f_2}{\partial y}| < 1$ if, in addition, $0 < |y| \leq r$," is not verified. The origin is the only fixed point of the system. Given an initial condition (x_0, y_0), the orbit converges to the periodic orbit of period 2, $(0, y_0 - (2/3)x_0)$, $(0, -y_0 + (2/3)x_0)$.



Example 2.

EXAMPLE 3. Let $f(x) = -x\sin^2 x$. Then $x = 0$ is the only fixed point of f and $f'(0) = 0$. Hence for x_0 sufficiently close to 0, the orbit $O(x_0)$ converges to 0. However, the condition $|f'(x)| \leq 1$ is violated and there is a periodic orbit of period 2 (see graph).



Example 3.

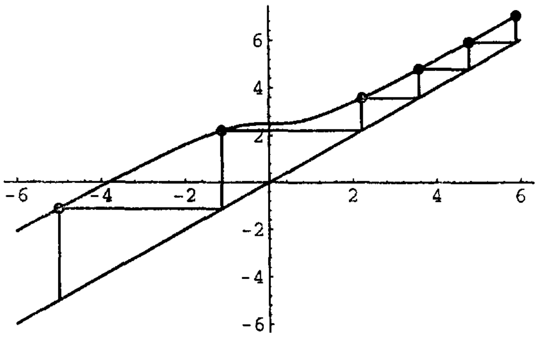
In all three theorems we assumed the existence of a fixed point. Example 4 shows what can happen when this assumption is deleted.

EXAMPLE 4. Let $f(x) = x - \arctan x + 2$. We have $0 \leq f'(x) < 1$ for all $x \in \mathbf{R}$. There are no fixed points of f and every orbit goes to $+\infty$.

We conclude this section and the paper with the following remark. Given a function $f : \mathbf{R} \to \mathbf{R}$ such that $|f'(x)| < 1$, there are only two alternatives:

- either f has a unique fixed point and every orbit converges to it (see Theorem 1), or
- every orbit goes to infinity.

In fact, in the case when no fixed point exists, the graph of f is always above or below the line



Example 4.

$y = x$. In the first case every orbit goes to $+\infty$ (see Example 4) and in the second case goes to $-\infty$. What can we say in higher dimension? If $\|F'(\mathbf{x})\| < 1$ and there is a unique equilibrium point then every orbit converges to it. This is easy to see. But if there is no equilibrium point? Is every orbit going to ∞? The answer is affirmative and its proof is based on the equality $F(L(\mathbf{x}_0)) = L(\mathbf{x}_0)$. In fact, assume that there is a bounded orbit $O(\mathbf{x}_0)$. Then $L(\mathbf{x}_0)$ is compact and there are two points $\mathbf{z}, \mathbf{w} \in \mathbf{L}(\mathbf{x}_0)$ such that

$$\|\mathbf{z} - \mathbf{w}\| = \max\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{x}, \mathbf{y} \in L(\mathbf{x}_0)\}.$$

The equality $F(L(\mathbf{x}_0)) = L(\mathbf{x}_0)$ implies the existence of $\mathbf{u}, \mathbf{v} \in L(\mathbf{x}_0)$ such that $F(\mathbf{u}) = \mathbf{z}$, $F(\mathbf{v}) = \mathbf{w}$. Then $\|\mathbf{z} - \mathbf{w}\| < \|\mathbf{u} - \mathbf{v}\|$, a contradiction. Therefore, under the assumption $\|F'(\mathbf{x})\| < 1$ we always have the alternative that either there is a unique fixed point and every orbit converges to it, or there is no fixed point and every orbit goes to infinity.

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